Journal of Mathematical Analysis and Applications **229,** 319–329 (1999)
Article ID jmaa.1998.6175, available online at http://www.idealibrary.com on IDEAL®

An Algebraic Approach to the Classification of Centers in Polynomial Liénard Systems

Colin Christopher

Submitted by U. Kirchgraber

Received October 31, 1997

We consider the second-order Liénard system

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where $f(x)$ and $g(x)$ are polynomials, which we rewrite in the equivalent two-dimensional form,

$$\dot{x} = y, \quad \dot{y} = -g(x) - yf(x).$$

We show that the local question of whether a critical point of this system is a center can be expressed in terms of global conditions on f and g. Using these results, we give a simple classification of all such centers. We also address the problem of the coexistence of centers, foci, and limit cycles for these systems. Systems with degenerate centers are also considered.

1. INTRODUCTION

We consider the system

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

where $f(x)$ and $g(x)$ are polynomials, which we rewrite in the equivalent two-dimensional form

$$\dot{x} = y, \quad \dot{y} = -g(x) - yf(x). \tag{2}$$

In [1], it was shown how a necessary condition for the existence of a center for the system (2) could be reduced to the calculation of a resultant of two polynomials. Here, we extend this result to obtain global conditions on

0022-247X/99 $30.00

the form of the polynomials f and g, which are both necessary and sufficient for a center. This new formulation allows us to classify the conditions for a center completely, as well as to tackle the question of the coexistence of centers with foci and limit cycles.

One of the surprising results of this work is the relative simplicity of these conditions. This is in sharp contrast to the topological behavior of these systems, which has only recently been classified when f and g are quadratic [6], or f is linear and g cubic [5, 7]. Even when g is linear, nothing general is known beyond these results.

This paper is part of a wider investigation into the algebraic implications of local integrability in polynomial systems, and in particular the existence of a center. This problem has been the focus of a lot of attention over the past century. Apart from its intrinsic interest, especially in bifurcation theory, it also highlights an important feature of polynomial systems not shared by more general analytic systems. We summarize a little of our knowlege below. More detail on algebraic integrability and some interesting historical notes can be found in [13, 14].

All nondegenerate centers for quadratic systems and for several large classes of cubic system have been classified and fall into two classes: those with an integrating factor of the form

$$e^{D/E} \prod C_i^{l_i},$$

where E, D, and the C_i are all polynomials, and those that arise from a simpler differential equation via a singular transformation.

Results of Singer [15] (after some modification [4]) show that any polynomial system whose first integral can be expressed in closed form with quadratures also falls into this first category.

Since the full cubic case is computationally intractable at present, any progress to higher degree systems must necessarily focus on significant subclasses. The results presented here show that all centers of polynomial Liénard systems (2) fall into the second category. It therefore provides the first such classification for a significant infinite dimensional class of systems.

Criteria for the local integrability of several larger classes of systems have been studied by Cherkas in [2, 3]. We shall show in a future paper how a complete classification into the categories above can be achieved for some of these systems.

In the case where f and g are nonanalytic, the classification of centers is more problematic. Indeed, even symmetry conditions need not imply that there is a center (take, for example, the case where $g = x$ and $f = \mu|x|$ for μ sufficiently large). Much work has been done on sufficient conditions for local and global centers in this context [8, 9, 11, 12, 16]. In several of these works, the symmetry (4) has been investigated in futher detail, and conditions such as $8g_{2r} < f_r^2$ mentioned in Section 4 are derived. A useful summary and references can be found in [9].

The paper proceeds as follows. A brief resumé of the work of Cherkas is given in Section 2, and the algebraic classification is derived in Section 3. Applications of our results to the center problem and the coexistence of centers and other features is considered in Section 4. We also consider how the results here can be adapted to degenerate centers following Moussu [10].

2. ANALYTIC PROPERTIES OF THE CRITICAL POINT

In this section we describe the analytic conditions for a center for the system (2) obtained by Cherkas in [1]. Since the derivation is simple, we give it here for completeness.

Any critical point of (2) lies on the x axis. Since a translation of the x axis will not destroy the form of the equation, we can assume without loss of generality that the critical point of interest is at the origin. The condition that this critical point should be nondegenerate and of focal type now implies that $g(0) = 0$ with $g'(0) > 0$. It also implies that $f(0)^2 < 4g'(0)$, however, we do not need this condition here.

We now wish to transform (2) into a more amenable form. Denote

$$F(x) = \int_0^x f(\xi)\,d\xi, \quad G(x) = \int_0^x g(\xi)\,d\xi.$$

Under the Liénard transformation $y \mapsto y + F(x)$, the system is brought to the form

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x). \tag{3}$$

We can simplify (3) further by a transformation that effectively removes g. Let u be the positive root of $2G$. From the conditions on g given above, it is clear that this root is well defined and analytic in a neighborhood of $x = 0$. Thus

$$u = (2G(x))^{1/2}\,\mathrm{sgn}(x) = (g'(0))^{1/2}x + O(x^2) \tag{4}$$

defines an invertible analytic transformation in a neighborhood of $x = 0$. Let $x(u)$ denote its inverse. The transformation takes the system (3) to the system

$$\dot{u} = \frac{g(x(u))}{u}\big(y - F(x(u))\big), \quad \dot{y} = -g(x(u)).$$

Since $g(x(u))/u = (g'(0))^{1/2} + O(u)$ is analytic and nonzero in a neighborhood of the origin, we can rescale (4) by multiplying the right-hand side by $u/g(x(u))$, which gives

$$\dot{u} = y - F(x(u)), \quad \dot{y} = -u. \tag{5}$$

This system has exactly the same direction field as (4) in a neighborhood of the origin, and hence the local qualitative behavior of the system, in particular, the existence of a center, is not altered by this scaling.

We write the power series for $F(x(u))$ as $\sum_1^\infty a_i u^i$. It turns out that the origin of (5) is a center if and only if all of the a_{2i+1} vanish. To see this we introduce the function $F^*(u) = \sum_1^\infty a_{2i} u^{2i}$, analytic in a neighborhood of the origin, and consider the system

$$\dot{u} = y - F^*(u), \quad \dot{y} = -u. \tag{5*}$$

Since the flow of (5*) is monodromic, it is clear that it must have a center at the origin because of symmetry in the u-axis. However, the system (5) is rotated with respect to (5*) in a neighborhood of the origin unless all of the terms a_{2i+1} vanish. Thus (5) cannot have a center at the origin unless all of the a_{2i+1} vanish. On the other hand, if all of the a_{2i+1} do vanish, then the system is a center by symmetry.

We can express this necessary and sufficient condition in a more geometrical form:

THEOREM 1. *The system* (2) *has a center at the origin if and only if* $F(x) = \Phi(G(x))$, *for some analytic function* Φ, *with* $\Phi(0) = 0$.

Proof. The argument above shows that there is a center if and only if $F(x(u)) = \phi(u^2)$ for some analytic function ϕ, $\phi(0) = 0$. But $u^2 = 2G(x)$, so set $\Phi(w) = \phi(2w)$.

Now consider the function $z(x)$ defined in a neighborhood of the origin by $z(x) = x(-u(x))$. We can also describe $z(x)$ as the unique analytic function that satisfies

$$G(x) = G(z), \qquad \big(z(0) = 0,\ z'(0) < 0\big).$$

That this equation defines a unique analytic function $z(x)$ is clear from the conditions on g, since

$$G(x) - G(z) = (x - z)\big(\tfrac{1}{2}g'(0)(x + z) + o(x, z)\big) = 0$$

has two analytic branches at the origin $z = x$ and $z = -x + o(x)$. The conditions on $z'(0)$ then select the second of these. Now $2G(x(u)) = u^2 = 2G(x(-u))$, whence $G(x) = G(x(-u(x)))$. Furthermore, $x(-u(x)) = -x + O(x^2)$; this solution must be $x(-u(x))$.

We know that the origin is a center if and only if the function $F(x(u))$ is even. That is, $F(x(u)) - F(x(-u))$ vanishes identically. But this is equivalent to saying that $F(x(u(x))) - F(x(-u(x))) = F(x) - F(z) = 0$. Thus, we have the following characterization of centers.

THEOREM 2. *The system* (1) *has a center at the origin if and only if there exists a function* $z(x)$ *satisfying*

$$F(x) = F(z), \quad G(x) = G(z), \qquad \big(z(0) = 0,\ z'(0) < 0\big). \tag{6}$$

This result also works when f and g are only analytic functions in a neighborhood of the origin. However, if f and g are also polynomials, then the solution $z(x)$ must correspond to a common factor between the functions $F(x) - F(z)$ and $G(x) - G(z)$ other than $(x - z)$. Thus, the following corollary is clear:

COROLLARY 3. *If the system* (2) *with* f *and* g *polynomials has a center at the origin*, *then it is necessary that the resultant of*

$$\frac{F(x) - F(z)}{x - z} \quad \text{and} \quad \frac{G(x) - G(z)}{x - z}$$

with respect to x *or* z *vanishes*. *This condition is sufficient if the common factor of the two polynomials vanishes at* $x = z = 0$.

3. ALGEBRAIC PROPERTIES

Corollary 3 gives algebraic conditions for a center but does not indicate how systems satisfying these conditions arise. In particular, if we want to look at families of systems with centers, we would like a condition that is not only more transparent, but hopefully reveals more of the symmetry underlying the existence of a center. The aim of this section is to derive a criterion that is both transparent and geometric.

Consider the subfield of $\mathbb{R}(x)$ generated by the polynomials F and G. Call this field $\mathcal{F}$. The field $\mathcal{F}$ shares an important property with F and G:

LEMMA 4. *Suppose there exists an analytic function* $z(x)$ *with* $z(0) = 0$, $z'(0) < 0$, *such that both* $F(z(x)) = F(x)$ *and* $G(z(x)) = G(x)$ *in a neighborhood of* $x = 0$. *Then for all elements* H *of the field* $\mathcal{F}$ *generated by* F *and* G, *we have* $H(z(x)) = H(x)$, *considered as meromorphic functions of* x *about* $x = 0$.

Proof. Note first that $H(z(x)) = 0$ if and only if $H(x) = 0$. Thus we need only verify that addition, multiplication and inversion of nonzero elements of $\mathcal{F}$ preserve this property, which is clearly the case.

Recall that Lüroth's Theorem states that if k is a field, any subfield of $k(x)$ that strictly contains k is isomorphic to $k(x)$. That is to say, the subfield is just $k(r)$ for some $r \in k(x)$. But $\mathcal{F}$ is a subfield of $\mathbb{R}(x)$ strictly containing $\mathbb{R}$, and so we must have $\mathcal{F} = \mathbb{R}(r)$ for some rational function $r \in \mathbb{R}(x)$.

Let us write r as A/B with $A, B \in \mathbb{R}[x]$. It is well known that the field generated over $\mathbb{R}$ by r can also be generated by

$$\frac{\alpha r + \beta}{\gamma r + \delta} = \frac{\alpha A + \beta B}{\gamma A + \delta B}$$

for any constants α, β, γ, and δ, such that $\alpha\delta - \beta\gamma \neq 0$. Since we can choose these constants to ensure that the degree of the denominator is less than the degree of the numerator, we can assume without loss of generality that B has degree less than A. We can also assume that all common factors between A and B are canceled and that B is monic.

Now F and G are in $\mathcal{F}$, and so

$$F = \frac{F_1(A, B)}{F_2(A, B)}, \quad G = \frac{G_1(A, B)}{G_2(A, B)},$$

where the F_i and G_i are homogeneous polynomials that we choose to have no common factors as polynomials in A and B. We now show the following:

LEMMA 5. $B = 1$.

Proof. We first factor the expressions for F_1 and F_2 over $\mathbb{C}[A, B]$ to obtain

$$F_1(A, B) = \prod_{i=1}^{r} (\lambda_1 A + \mu_i B), \quad F_2(A, B) = \prod_{i=r+1}^{r+s} (\lambda_1 A + \mu_i B),$$

for some complex constants λ_i and μ_i.

Now note that if $\lambda_1 A + \mu_1 B$ and $\lambda_2 A + \mu_2 B$ have a common factor as polynomials in x, then they are multiples of each other, since A and B have no common factors in x (over $\mathbb{R}$ and therefore over $\mathbb{C}$, too). However, we chose F_1 and F_2 to have no common factors as polynomials in A and B, hence they must have no common factors as polynomials in x.

Thus the denominator of F as a rational function of x after cancellation with the numerator is just $F_2(A(x), B(x))$, and so

$$\prod_{i=r+1}^{r+s} (\lambda_1 A + \mu_i B) \in \mathbb{R}.$$

Since the degree of A is larger than the degree of B, this can only happen when $\lambda_i = 0$ for all $i = r+1, \ldots, s$, and hence B is must be a constant polynomial and therefore equal to 1. Similar considerations show that $G_2(A(x), B(x))$ is also a constant.

Thus we have shown that both F and G are polynomials of some polynomial $A \in \mathcal{F}$. The final step follows.

THEOREM 6. *The system* (2) *with* $g(0) = 0$ *and* $g'(0) > 0$ *has a nondegenerate center at the origin if and only if* $F(x)$ *and* $G(x)$ *are both polynomials of a polynomial* $A(x)$ *with* $A'(0) = 0$ *and* $A''(0) \neq 0$.

Proof. By Theorem 2, if there is a center at the origin of (2), then there is a function $z(x)$ with $z(0) = 0$ and $z'(0) < 0$ such that $F(z(x)) = F(x)$ and $G(z(x)) = G(x)$. By Lemma 4, the polynomial generator of $\mathcal{F}$, A also satisfies $A(z(x)) = A(x)$, and hence its linear term must vanish. Now $G(x)$ is a polynomial in A with $G''(0) > 0$, which means that the quadratic term of A cannot vanish.

Conversely, assume F and G are polynomials of a polynomial A with a nonzero quadratic term but no linear term. From the conditions on A, we can find an analytic function satisfying $A(z(x)) = A(x)$ with $z(0) = 0$, $z'(0) < 0$. Clearly F and G must then satisfy condition (6) of Theorem 2, and the origin is therefore a center.

COROLLARY 7. *The system* (2) *has a nondegenerate center at the point* $x = p$ *if and only* $g(p) = 0$, $g'(p) > 0$, *and* F *and* G *are both polynomials of a polynomial* A *that satisfies* $A'(p) = 0$ *with* $A''(p) \neq 0$.

Proof. If we shift the x axis to bring $x = p$ to the origin, then it is clear that the new F and G calculated will differ from the original ones only by a constant. The rest follows quite easily from Theorem 6.

The following corollary also follows directly from Theorem 6.

COROLLARY 8. *A necessary condition for the system* (2) *to have a center at some point is that* $\gcd(i + 1, j + 1) > 1$.

4. APPLICATIONS AND FINAL REMARKS

We wish to consider some applications of our classification in this section. First, it would be helpful to attach some geometrical meaning to the criteria given in Theorems 2 and 6.

THEOREM 9a. *If the system* (2) *has a nondegenerate center at the origin, then the transformation* $x \mapsto z(x)$, *given from the conditions of Theorem* 2, *takes the direction field of* (2) *into itself, reversing the directions. Thus the origin has a generalized symmetry.*

THEOREM 9b. *Alternatively, under the same conditions the system can be obtained from a system of lower degree*:

$$\dot{w} = y, \quad \dot{y} = -m(w) - l(w)y, \tag{7}$$

via a singular transformation $w \mapsto h(x)$ *for some polynomial* $h(x) = x^2 + O(x^3)$, *and a singular scaling. Here* l *and* m *are polynomials* $m(0) > 0$, *and the transformation* $v \mapsto h(x)$ *takes a noncritical point at the origin of* (7) *and "unfolds" it into a center.*

Remark. A more precise way to describe this unfolding operation would be to take the pullback of the associated 1-forms.

Proof. The first assertion is a direct calculation from condition (6) of Theorem 2. The generalized symmetry condition means that trajectories lying in $x \geq 0$ can be mapped onto trajectories in $x \leq 0$, with the points on $x = 0$ being fixed. If we know that the flow encircles the origin, then trajectories sufficiently close to the origin must be closed. Thus if the critical point is known to be of focal type, this generalized symmetry is enough to imply the existence of a center.

For the second part, we take the polynomial A of Theorem 6 and consider

$$h(x) = 2\frac{A(x) - A(0)}{A''(0)} = x^2 + O(x^3).$$

Clearly F and G are also polynomials of this polynomial, so that $F = L(h(x))$ and $G = M(h(x))$ for some polynomials L and M. From the condition on $g'(0)$, we see that $M'(0) > 0$. Take $l = L'$ and $m = M'$, then system (7) transforms (after scaling by $h'(x)$) to

$$\dot{x} = y, \quad \dot{y} = -h'(x)m(h(x)) - h'(x)l(h(x))y = -g(x) - f(x)y.$$

The origin of (7) is not a critical point, but locally the trajectories are of the form

$$w = \alpha - \frac{1}{2m(\alpha)}y^2 + O(y^3)$$

for small values of α, where the $O(y^3)$ term is analytic in α as well as y. The transformation takes these trajectories to the curves

$$x^2 + O(x^3) = \alpha - \frac{1}{2m(\alpha)}y^2 + O(y^3),$$

for α sufficiently small. These trajectories are thus closed curves approximating to the ellipses $x^2 + y^2/(2m(0)) = \alpha$, and the origin is a center.

Another application of the results of Theorem 6 is to provide an easy demonstration of the coexistence of foci, limit cycles, and centers in different parts of the phase plane for certain systems of the form (2).

THEOREM 10. *There exist polynomial Liénard systems with coexisting centers and foci, coexisting centers and fine foci, and coexisting limit cycles and centers.*

Proof. Take the following family of vector fields:

$$\dot{x} = y, \quad \dot{y} = -(1/8 + M/4)\frac{dM}{dx} - y\big(1 + (2\lambda - 4)M^2\big)\frac{dM}{dx},$$

where $M = (x^2 - 2x^3)/2$. It is a simple matter to establish that this system has a center at the origin, by an application of Corollary 7. Furthermore, for small $\lambda \neq 0$, the critical point at (1,0) is a focus with divergence λ and becomes a stable fine focus of order 1 when $\lambda = 0$. Thus there is a limit cycle in a neighborhood of (1,0) for small positive λ by the Hopf bifurcation theorem.

The role of Theorem 6 in this investigation is not only to prove the existence of a center, but also to single out effectively the relevant class of high-degree systems (here degree 8) in which to search. The result is something that is manageable even by a hand calculation.

In conclusion, we sketch how the results in the first two sections of this paper can be extended to cover the case when system (2) has a degenerate center at the origin, that is, when $g(0) = g'(0) = 0$. We define F and G as before.

From Moussu [10], the existence of a monodromic flow about the origin is equivalent to the absence of separatrices at the origin of (2). In turn, a necessary and sufficient condition for the absence of separatrices is that

$$G(x) = g_{2r}x^{2r} + O(x^{2r+1}), \quad F(x) = f_r x^r + O(x^{r+1}),$$

for some $r > 0$ and that $8g_{2r} > f_r^2$. Here f_r can vanish, but the inequality forces $g_{2r} > 0$.

To make the comparison with the nondegenerate case clear, we shall assume that the origin of (2) is already known to be monodromic. That is, we replace the hypothesis that $g(0) = 0$, $g'(0) > 0$, in the statement of the theorems with the conditions

$$G = g_{2r}x^{2r} + O(x^{2r+1}), \quad F = f_r x^r + O(x^{r+1}), \qquad 8g_{2r} > f_r^2. \tag{8}$$

Instead of the transformation (4), we now take

$$u^{2r} = 2rG(x) = 2rg_{2r}x^{2r} + O(x^{2r+1}) \quad \big(u(0) = 0, u'(0) > 0\big). \tag{4'}$$

This defines u as an analytic function of x. Using this new variable, we obtain the system.

$$\dot{u} = y - F(x(u)), \quad \dot{y} = -u^{2r-1}. \tag{5'}$$

Here we have scaled the right-hand side of the equation by $u^{2r-1}/g(x(u))$, which is well defined and nonzero near $u = 0$. From Moussu [10], this is a center if and only if $F(x(u)) = F(x(-u))$.

It is clear that Theorem 1 no longer holds, except in the form $F(x(u)) = F(x(-u))$, which just restates Moussu's criterion. However, in Theorem 2, the condition $G(x) = G(z)$ still defines a unique analytic function $z(x)$ with $z'(0) < 0$. A similar argument shows that $z(x) = x(-u(x))$, and hence $F(x) = F(z)$ if and only if $F(x(u)) = F(x(-u))$. Theorem 2 thus remains true.

Corollary 3, Lemma 4, and Lemma 5 carry through without change. Hence in Theorem 6 it is still a necessary condition that F and G are polynomials of a polynomial A, and that $A(x) = A(z(x))$. Since $z'(0) < 0$, this implies that $A'(0) = 0$, and that $A = a_{2q}x^{2q} + O(x^{2q+1})$ for some $q > 0$, with $a_{2q} \neq 0$. However, the condition on G only implies that $2q$ must divide $2r$ (and not $A''(0) \neq 0$, as before).

Conversely, if F and G verify the conditions (8) and are polynomials of a polynomial $A = a_{2q}x^{2q} + O(x^{2q+1})$, $q > 0$ with $a_{2q} \neq 0$, then $A(x) = A(z(x))$ defines a function $z(x)$ satisfying the conditions of Theorem 2, and so the origin is a center. Corollary 7 can be adapted similarly, and Corollary 8 follows directly from Theorem 6.

Finally, Theorem 9a remains unchanged. If the critical point is assumed to be mondromic, then the generalized symmetry will imply a center. However, Theorem 9b must be adapted slightly. The center still arises from a singular transformation of the type described, but the simpler system (7) may also have a critical point at the origin. If this does occur, then it is easy to see that this critical point is also monodromic by Moussu's criteria, and so the singular transformation still gives a center. Further details are left to the reader.

We note in passing that a classification similar to the one in this paper can be performed for complex Liénard systems with a nondegenerate critical point with equal and opposite eigenvalues. This case includes the center case of this paper, as well as the case of an integrable saddle.

ACKNOWLEDGMENTS

I thank Dana Schlomiuk for helpful comments on an earlier draft of this paper, and the referee for bringing to my attention the information and references on the nonanalytic cases mentioned above.

REFERENCES

1. L. A. Cherkas, On the conditions for a center for certain equations of the form $yy' = P(x) + Q(x)y + R(x)y^2$, *Differential Equations* **8**(8) (1974), 1104–1107.
2. L. A. Cherkas, Conditions for a Liénard equation to have a center, *Differential Equations* **12**(2) (1977), 201–206.

3. L. A. Cherkas, Conditions for the equation $yy' = \sum_{i=0}^{3} P_i(x)y^i$ to have a center, *Differential Equations* **14**(9) (1979), 1133–1137.
4. C. J. Christopher, Liouvillian first integrals of second order polynomial differential equations, preprint, University of Plymouth (1997).
5. F. Dumortier and C. Li, On the uniqueness of limit cycles surrounding one or more singularities for Liénard equations, *Nonlinearity* **9** (1996), 1489–1500.
6. F. Dumortier and C. Li, Quadratic Liénard equations with quadratic damping, *J. Differential Equations* **139** (1997), 41–59.
7. F. Dumortier and C. Rousseau, Cubic Liénard equations with linear damping, *Nonlinearity* **3** (1990), 1015–1039.
8. A. F. Filippov, Sufficient conditions for the existence of stable limit cycles of second order equations, *Mat. Sb.* **30** (72) (1952), 171–180 (in Russian).
9. T. Hara and T. Yoneyama, On the global center of generalised Liénard equation and its application to stability problems, *Funcial. Ekvac.* **28** (1985), 171–192.
10. R. Moussu, Symétrie et forme normale des centers et foyers dégénérés, *Ergodic Theory Dynam. Systems* **2** (1982), 241–251.
11. Z. Opial, Sur un théorèm de A. Filippoff, *Ann. Polon. Math.* **5** (1958), 67–75.
12. G. Sansoni and R. Conti, “Non-linear Differential Equations,” Macmillan, New York, 1964.
13. D. Schlomiuk, Algebraic particular integrals, integrability and the problem of the center, *Trans. Amer. Math. Soc.* **338**(2) (1993), 799–841.
14. D. Schlomiuk, Algebraic and geometric aspects of the theory of polynomial vector fields, *in* “Bifurcations and Periodic Orbits of Vector Fields” (D. Schlomiuk, ed.), pp. 429–467, Kluwer Academic, Dordrecht, The Netherlands, 1993.
15. M. F. Singer, Liouvillian first integrals of differential equations, *Trans. Amer. Math. Soc.* **333**(2) (1992), 673–688.
16. J. Sugie and T. Hara, Classification of global phase portraits of a system of Liénard type, *J. Math. Anal. Appl.* **193**(1) (1995), 264–281.